On highly potential words

Bojan Bašić

Department of Mathematics and Informatics, University of Novi Sad,

Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

bojan.basic@dmi.uns.ac.rs

Abstract

We introduce a class of infinite words, called *highly potential words* because of their seemingly high potential of being a good supply of examples and counterexamples regarding various problems on words. We prove that they are all aperiodic words of finite positive defect, and having their set of factors closed under reversal, thus giving examples Brlek and Reutenauer were looking for. We prove that they indeed satisfy the Brlek-Reutenauer conjecture. We observe that each highly potential word is recurrent, but not uniformly recurrent. Considering a theorem from the paper of Balková, Pelantová and Starosta, later found to be incorrect, we show that highly potential words constitute an infinite family of counterexamples to that theorem. Finally, we construct a highly potential word which is a fixed point of a nonidentical morphism, thus showing that a stronger version of a conjecture by Blondin-Massé et al., as stated by Brlek and Reutenauer, is false.

Mathematics Subject Classification (2010): 68R15

Keywords: palindrome, palindromic complexity, factor complexity, word defect

1 Introduction

Problems on palindromes in infinite words have recently been studied quite a lot. Some new notions were introduced, such as palindromic defect and palindromic complexity, and Brlek and Reutenauer [8] conjectured an equality stating a connection between the defect, the palindromic complexity, and the (factor) complexity of an infinite word w whose set of factors is closed under reversal. They proved their conjecture for periodic words, and observed that, on the basis of some earlier results, the conjecture also holds for words of defect 0. They further tested the conjecture for some words of infinite defect, namely: the Thue-Morse word, the paper folding sequences and the generalized Rudin-Shapiro sequences, and the result was positive. The next logical step would be to find more evidence for the conjecture by testing it for aperiodic words of finite positive defect, at least for a few examples, but it turned out that the authors were unable to find even a single such word (having the set of factors closed under reversal).

In the same paper, Brlek and Reutenauer recalled the conjecture of Blondin-Massé et al. [6], stating that there does not exist an aperiodic word of finite positive defect that is a fixed point of some primitive morphism. Under the stronger conjecture that there does not exist an aperiodic word of finite positive defect that is a fixed point of any nonidentical morphism, Brlek and Reutenauer showed that their conjecture holds for fixed points of nonidentical morphisms. The assumed conjecture remained open.

Balková, Pelantová and Starosta [3] proved the Brlek-Reutenauer conjecture for uniformly recurrent words. Apart from this proof, they gave a few other related theorems, one of which was later shown to be incorrect [5]. However, only one counterexample has been found, having a rather pathological flavor. It remained unanswered whether there are more counterexamples, possibly constituting some less artificial family.

In this paper we introduce a class of words related to all the problems discussed above. Namely, the work is divided into sections as follows.

In Section 2 we recall the necessary definitions and results.

In Section 3 we define a construction of a class of words. Since they seem to have a high potential to serve as examples and counterexamples in various problems on words, we dub them *highly potential words*. We observe that each highly potential word has its set of factors closed under reversal, that it is aperiodic, recurrent, but not uniformly recurrent. We prove that each highly potential word has finite positive defect.

In Section 4 we prove that the Brlek-Reutenauer conjecture indeed holds for highly potential words. Note that, since highly potential words are not uniformly recurrent, this result does not follow from the result of Balková, Pelantová and Starosta.

In Section 5 we show that highly potential words are counterexamples to the statement of a theorem by Balková, Pelantová and Starosta.

In Section 6 we construct a highly potential word that is a fixed point of a nonidentical morphism. Since highly potential words are aperiodic words of finite positive defect, this construction disproves Brlek and Reutenauer's strengthening of the conjecture by Blondin-Massé et al.

2 Preliminaries

Given a finite alphabet Σ, let Σ^* denote the set of all finite words over Σ, and let Σ^∞ denote the set of all finite or infinite words over Σ (that is, the set of all finite or infinite sequences of letters from Σ). The length of a word $w \in \Sigma^*$ is denoted by $|w|$, and the unique word of length equal to 0, called the *empty word*, is denoted by ε. Given a finite word w and $k \in \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of nonnegative integers, while we reserve the notation $\mathbb{N}$ for positive integers), we write w^k for $\underbrace{ww\dots w}_{k \text{ times}}$ (called the *kth power* of a word w). We write w^∞ for the infinite word $www\dots$. Infinite words that are of the form w^∞ are called *periodic*. Infinite words that are of the form uw^∞ (where $u, w \in \Sigma^*$, $w \neq \varepsilon$) are called *eventually* (or *ultimately*) *periodic*. Infinite words that are not eventually periodic are called *aperiodic*.

We define map $\widetilde{\ }\colon \Sigma^* \to \Sigma^*$, called *reversal*, as follows: if $w = a_1a_2\dots a_n$, where $a_1, a_2, \dots, a_n \in \Sigma$, then $\widetilde{w} = a_na_{n-1}\dots a_1$. A word w is a *palindrome* if $w = \widetilde{w}$.

A word $v \in \Sigma^*$ is a *suffix* of a word $w \in \Sigma^*$ if there exists a word $u \in \Sigma^*$ such that $w = uv$. A word $v \in \Sigma^*$ is a *prefix* of a word $w \in \Sigma^\infty$ (resp. $w \in \Sigma^*$) if there exists a word $u \in \Sigma^\infty$ (resp. $u \in \Sigma^*$) such that $w = vu$. A word $v \in \Sigma^*$ is a *factor* of a word $w \in \Sigma^\infty$ (resp. $w \in \Sigma^*$) if there exist words $u_1 \in \Sigma^*$, $u_2 \in \Sigma^\infty$ (resp. $u_2 \in \Sigma^*$) such that $w = u_1vu_2$. The set of all factors of a word w is denoted by $\mathrm{Fact}(w)$. We say that the set of factors of w is *closed under reversal* if for any $v \in \mathrm{Fact}(w)$ we have $\widetilde{v} \in \mathrm{Fact}(w)$. The set of all palindromic factors of a word w is denoted by $\mathrm{Pal}(w)$.

An infinite word w is *recurrent* if each of its factors has infinitely many occurrences in w, and w is *uniformly recurrent* if it is recurrent and, for each of its factors, the gaps between consecutive occurrences of it in w are bounded (by *gap*, we mean the difference between two positions at which two consecutive occurrences of the considered factor begin). The following theorems, the proof of which can be found, e.g., in [10, Proposition 2.11] and [1, Example 10.9.1], respectively, will be useful.

Theorem 2.1. *Given an infinite word w, if* $\mathrm{Fact}(w)$ *is closed under reversal, then w is recurrent.*

Theorem 2.2. *If an infinite word is periodic, then it is uniformly recurrent.*

A function $\varphi : \Sigma^* \to \Sigma^*$ is called a *morphism* if for all $w, v \in \Sigma^*$ we have $\varphi(wv) = \varphi(w)\varphi(v)$. Clearly, a morphism is uniquely determined by images of the letters, and thus it is possible to extend any given morphism to infinite words in the natural way. We say that a word $w \in \Sigma^\infty$ is a *fixed point* of a morphism φ if $\varphi(w) = w$.

Let an infinite word w be given. The *factor complexity* (or only *complexity*) of w is the function $C_w : \mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$C_w(n) = |\{v \in \mathrm{Fact}(w) : |v| = n\}|.$$

The *palindromic complexity* of w is the function $P_w : \mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$P_w(n) = |\{v \in \mathrm{Pal}(w) : |v| = n\}|.$$

Of course, we have

$$|\mathrm{Pal}(w)| = \sum_{n=0}^{|w|} P_w(n).$$

We now recall an inequality due to Droubay, Justin and Pirillo [9, Proposition 2].

Theorem 2.3. *For any finite word w we have:*

$$|\mathrm{Pal}(w)| \leqslant |w| + 1.$$

This inequality motivated Brlek et al. [7] to define *palindromic defect* (or only *defect*) of a finite word w by

$$D(w) = |w| + 1 - |\mathrm{Pal}(w)|.$$

The following theorem (see, e.g., [6, Lemma 1]) gives an important property of the defect.

Theorem 2.4. *Let $w \in \Sigma^*$ and $v \in \mathrm{Fact}(w)$. Then $D(v) \leqslant D(w)$.*

The above theorem motivates the following definition of the defect of an infinite word w:

$$D(w) = \sup_{v \in \mathrm{Fact}(w)} D(v).$$

Clearly, this equality also holds for finite words.

Another important inequality connecting the notions discussed above is proved by Baláži, Masáková and Pelantová [2, Theorem 1.2(ii)]:

Theorem 2.5. *Let w be an infinite word with $\mathrm{Fact}(w)$ being closed under reversal. For each $n \in \mathbb{N}_0$ we have*

$$P_w(n) + P_w(n+1) \leqslant C_w(n+1) - C_w(n) + 2.$$

Actually, in [2], the above inequality is formulated only for uniformly recurrent words. However, it is easy to check that the proposed proof never relies on this assumption, but only on the assumption that w is recurrent; since, by Theorem 2.1, any word w having $\mathrm{Fact}(w)$ closed under reversal is recurrent, the given statement follows.

Finally, we state the Brlek-Reutenauer conjecture, recounted in Section 1. It predicts the following equality dealing with the defect $D(w)$ and the function $T_w : \mathbb{N}_0 \to \mathbb{N}_0$, inspired by Theorem 2.5, defined by

$$T_w(n) = C_w(n+1) - C_w(n) + 2 - P_w(n) - P_w(n+1).$$

Conjecture 2.6. *Let w be an infinite word with $\mathrm{Fact}(w)$ being closed under reversal. Then:*

$$2D(w) = \sum_{n=0}^{\infty} T_w(n).$$

3 Highly potential words. Construction and basic properties

Let w be a finite word that is not a palindrome, and let c be a letter that does not occur in w. Define $w_0 = w$ and, for $i \in \mathbb{N}$,

$$w_i = w_{i-1} c^i \widetilde{w_{i-1}}. \tag{1}$$

Finally, let

$$\mathrm{hpw}(w) = \lim_{i \to \infty} w_i. \tag{2}$$

The meaning of the above limit is clear since each w_i is a prefix of w_{i+1}. We call $\mathrm{hpw}(w)$ the *highly potential word generated by w*.

Note. The above construction bears a resemblance (at least visually) to the Zimin words, introduced in [12] with a motivation in semigroup theory. The sequence of Zimin words is defined recursively by $Z_1 = a_1$ and $Z_{n+1} = Z_n a_{n+1} Z_n$, where $a_1, a_2, a_3 \ldots$ are letters. Somewhat recently, the Zimin words have been reinvented under the name Fraenkel words, related to a combinatorial translation of a problem from number theory (see, e.g., [11]). However, there are some major differences between highly potential words and the Zimin words (actually, not the Zimin words themselves but their limit instead, since that is what "corresponds" to a highly potential word):

1. The limit of Zimin words is only one object, while the class of highly potential words contains infinitely many infinite words; even more, each nonpalindromic finite word generates one highly potential word.

2. The main problem with the limit of Zimin words, at least from a perspective of studying infinite words, is the fact that the limit of Zimin words has infinitely many different letters. Therefore, it is even not an infinite word (over a finite alphabet). On the other hand, a highly potential word can have as few as three different letters, $\mathrm{hpw}(ab)$ being the example.

We also note the following interesting link between the Zimin words and highly potential words: if we note lengths of the sequences of consecutive occurrences of the letter c in $\mathrm{hpw}(w)$, we get:

$$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5 \ldots,$$

which is exactly the limit of sequence of Zimin words 1, 121, 1213121, 121312141213121...

The following proposition is easy to prove, but is of key importance.

Proposition 3.1. *Let* $\mathrm{hpw}(w)$ *be a highly potential word. Then:*

a) $\mathrm{Fact}(\mathrm{hpw}(w))$ *is closed under reversal;*

b) $\mathrm{hpw}(w)$ *is recurrent;*

c) $\mathrm{hpw}(w)$ *is not uniformly recurrent;*

d) $\mathrm{hpw}(w)$ *is aperiodic.*

Proof. a) Let $v \in \operatorname{Fact}(\operatorname{hpw}(w))$. Choose $i \in \mathbb{N}_0$ large enough such that $v \in \operatorname{Fact}(w_i)$. Since $w_{i+1} = w_i c^{i+1} \widetilde{w_i}$, we have $\tilde{v} \in \operatorname{Fact}(w_{i+1})$, and thus $\tilde{v} \in \operatorname{Fact}(\operatorname{hpw}(w))$.

b) Follows from a) and Theorem 2.1.

c) Since $w \in \operatorname{Fact}(\operatorname{hpw}(w))$, and since we can always find two consecutive occurrences of w in $\operatorname{hpw}(w)$ with arbitrarily many letters c in the gap between, the statement follows.

d) As in c), we see that not only that $\operatorname{hpw}(w)$ is not uniformly recurrent, but for any u, v such that $\operatorname{hpw}(w) = uv$, v is not uniformly recurrent; therefore, v is not periodic, which implies that $\operatorname{hpw}(w)$ is not eventually periodic, and the conclusion follows by Theorem 2.2. ∎

The main result of this section is the following theorem.

Theorem 3.2. *Let* $\operatorname{hpw}(w)$ *be a highly potential word. Then* $D(\operatorname{hpw}(w)) = D(w) + 1$. *In particular,*

$$0 < D(\operatorname{hpw}(w)) < \infty.$$

Proof. Let $w = w_0 = a_1 a_2 \ldots a_l$, where $a_1, a_2, \ldots, a_l \in \Sigma$. Since

$$w_1 = w_0 c \widetilde{w_0} = a_1 a_2 \ldots a_l c a_l \ldots a_2 a_1,$$

it is easy to see that

$$\operatorname{Pal}(w_1) = \operatorname{Pal}(w_0) \cup \{a_s a_{s+1} \ldots a_l c a_l \ldots a_{s+1} a_s : 1 \leqslant s \leqslant l\} \cup \{c\},$$

where these sets are disjoint. Therefore,

$$\begin{aligned} D(w_1) &= |w_1| + 1 - |\operatorname{Pal}(w_1)| = (2l+1) + 1 - (|\operatorname{Pal}(w_0)| + l + 1) \\ &= l + 1 - |\operatorname{Pal}(w_0)| = |w_0| + 1 - |\operatorname{Pal}(w_0)| = D(w_0) = D(w). \end{aligned}$$

Since

$$w_2 = w_1 cc \widetilde{w_1} = a_1 a_2 \ldots a_l c a_l \ldots a_2 a_1 cc a_1 a_2 \ldots a_l c a_l \ldots a_2 a_1,$$

having in mind that $w = a_1 a_2 \ldots a_l$ is not a palindrome, it is easy to see that

$$\begin{aligned} \operatorname{Pal}(w_2) = \operatorname{Pal}(w_1) & \\ &\cup \{a_s a_{s+1} \ldots a_l c a_l \ldots a_2 a_1 cc a_1 a_2 \ldots a_l c a_l \ldots a_{s+1} a_s : 1 \leqslant s \leqslant l\} \\ &\cup \{a_s \ldots a_2 a_1 cc a_1 a_2 \ldots a_s : l \geqslant s \geqslant 1\} \\ &\cup \{c a_l \ldots a_2 a_1 cc a_1 a_2 \ldots a_l c, cc\}. \end{aligned}$$

Therefore,

$$\begin{aligned} D(w_2) &= |w_2| + 1 - |\mathrm{Pal}(w_2)| = (4l+4) + 1 - (|\mathrm{Pal}(w_1)| + l + l + 2) \\ &= 2l + 3 - |\mathrm{Pal}(w_1)| = |w_1| + 2 - |\mathrm{Pal}(w_1)| = D(w_1) + 1 \\ &= D(w) + 1. \end{aligned}$$

We are now going to prove that $D(w_i) = D(w_{i-1})$ for each $i \geqslant 3$. Note that w_i is a palindrome for each $i \geqslant 1$. Let $i \geqslant 3$ be given, and let $w_{i-2} = b_1 b_2 \dots b_m$, where $b_1, b_2, \dots, b_m \in \Sigma$. We have

$$w_{i-1} = w_{i-2} c^{i-1} \widetilde{w_{i-2}} = b_1 b_2 \dots b_m c^{i-1} b_m \dots b_2 b_1$$

and

$$w_i = w_{i-1} c^i \widetilde{w_{i-1}} = b_1 b_2 \dots b_m c^{i-1} b_m \dots b_2 b_1 c^i b_1 b_2 \dots b_m c^{i-1} b_m \dots b_2 b_1.$$

We claim that

$$\begin{aligned} \mathrm{Pal}(w_i) = \mathrm{Pal}(w_{i-1}) & \\ & \cup \{b_s b_{s+1} \dots b_m c^{i-1} b_m \dots b_2 b_1 c^i b_1 b_2 \dots b_m c^{i-1} b_m \dots b_{s+1} b_s \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad : 1 \leqslant s \leqslant m\} \\ & \cup \{c^s b_m \dots b_2 b_1 c^i b_1 b_2 \dots b_m c^s : i-1 \geqslant s \geqslant 1\} \\ & \cup \{b_s \dots b_2 b_1 c^i b_1 b_2 \dots b_s : m \geqslant s \geqslant 1\} \\ & \cup \{c^i\} \cup \{c^s b_1 b_2 \dots b_m c^s : 1 \leqslant s \leqslant i-1\}. \end{aligned}$$

Indeed: all the palindromes added in the first set are new, because there is no factor c^i in w_{i-1}; all the palindromes added in the second, the third and the fourth set are new for the same reason; finally, all the words added in the fifth set are palindromes because $w_{i-2} = b_1 b_2 \dots b_m$ is a palindrome, and it can be seen that all of them also are new. Further, it can be easily checked that the list above is complete. Therefore,

$$\begin{aligned} D(w_i) &= |w_i| + 1 - |\mathrm{Pal}(w_i)| \\ &= (4m + 3i - 2) + 1 - (|\mathrm{Pal}(w_{i-1})| + m + (i-1) + m + 1 + (i-1)) \\ &= 2m + i - |\mathrm{Pal}(w_{i-1})| = |w_{i-1}| + 1 - |\mathrm{Pal}(w_{i-1})| = D(w_{i-1}). \end{aligned}$$

Altogether, $D(w_0) = D(w_1) = D(w)$ and $D(w_i) = D(w) + 1$ for $i \geqslant 2$. Because of Theorem 2.4 and the equality (2), we have:

$$\sup_{v \in \mathrm{Fact}(\mathrm{hpw}(w))} D(v) = \sup_{i \in \mathbb{N}_0} D(w_i),$$

and thus

$$D(\mathrm{hpw}(w)) = \sup_{i \in \mathbb{N}_0} D(w_i) = D(w) + 1,$$

which was to be proved. ∎

4 Conjecture 2.6 for highly potential words

In this section we prove that highly potential words satisfy the Brlek-Reutenauer conjecture.

Theorem 4.1. *For each highly potential word* $\mathrm{hpw}(w)$ *we have:*

$$2D(\mathrm{hpw}(w)) = \sum_{n=0}^{\infty} T_{\mathrm{hpw}(w)}(n).$$

The proof is preceded by a series of lemmas. Lemmas 4.2 and 4.3 are intermediate steps toward Lemmas 4.4 and 4.5, which are then used directly in the proof of Theorem 4.1. For the rest of this section, let $w = w_0 = a_1 a_2 \dots a_l$, where $a_1, a_2, \dots, a_l \in \Sigma$, and let $|w_i| = l_i$. Since, by (1), the sequence $l_0, l_1, l_2 \dots$ satisfies the recurrent relation $l_i = 2l_{i-1} + i$ with $l_0 = l$, it is an easy exercise in recurrent relations to show that $l_i = (l+2) \cdot 2^i - i - 2$.

Lemma 4.2. *Let $n \geqslant 1$ be given. Each $v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w)$ such that $|v| = n$ is uniquely determined by the number of consecutive occurrences of the letter c in the middle of the palindrome v.*

Further, the letter c may consecutively occur exactly $k \geqslant 1$ times in the middle of the palindrome v if and only if $k \leqslant n \leqslant (l+2) \cdot 2^k + k$ and $k \equiv n \pmod 2$.

Proof. Let us first show that for each $v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w)$ there is a sequence of consecutive occurrences of the letter c in the middle of the palindrome v. Suppose the opposite: the middle letter, or the two middle letters, of v are $\neq c$. Since $v \notin \mathrm{Pal}(w)$, it follows that there exists a letter c in the word v. We now have $v = ucv'c\tilde{u}$, where v' does not contain the letter c. However, from (1) easily follows that $v' = w_0$ or $v' = \widetilde{w_0}$, which is impossible since v' must be a palindrome while w_0 is not.

From (1), we see that the letter c occurs exactly k times consecutively only in the word w_k and its further copies in $\mathrm{hpw}(w)$. We have:

$$w_k = w_{k-1}c^k\widetilde{w_{k-1}};$$

$$w_{k+1} = w_k c^{k+1}\widetilde{w_k} = w_k c^{k+1} w_k = w_{k-1}c^k\widetilde{w_{k-1}}c^{k+1}w_{k-1}c^k\widetilde{w_{k-1}};$$

$$\begin{aligned} w_{k+2} &= w_{k+1}c^{k+2}\widetilde{w_{k+1}} = w_{k+1}c^{k+2}w_{k+1} \\ &= w_{k-1}c^k\widetilde{w_{k-1}}c^{k+1}w_{k-1}c^k\widetilde{w_{k-1}}c^{k+2}w_{k-1}c^k\widetilde{w_{k-1}}c^{k+1}w_{k-1}c^k\widetilde{w_{k-1}}. \end{aligned}$$

Therefore, the simultaneous "extending" of both ends of the word c^k can last only till we reach $c^{k+1}w_{k-1}c^k\widetilde{w_{k-1}}c^{k+1}$, since at this point the following letter on the right side is c and on the left side is $\neq c$, or vice versa (clearly, the same holds for further copies of w_{k+2}). Thus, the letter c consecutively occurs exactly k times in the middle of a palindrome of a given length n if and only if the palindrome is a middle section of $c^{k+1}w_{k-1}c^k\widetilde{w_{k-1}}c^{k+1}$, and therefore it is uniquely determined. Furthermore, we see that such a palindrome exists if and only if $k \equiv n \pmod 2$ and

$$\begin{aligned} k \leqslant n \leqslant 2l_{k-1} + 3k + 2 &= 2((l+2)\cdot 2^{k-1} - (k-1) - 2) + 3k + 2 \\ &= (l+2)\cdot 2^k - 2k + 2 - 4 + 3k + 2 \\ &= (l+2)\cdot 2^k + k, \end{aligned}$$

which was to be proved. ∎

Lemma 4.3. *Let $n \geqslant l+3$ be given. For each $v \in \mathrm{Fact}(\mathrm{hpw}(w))$ such that $|v| = n$, there either exists exactly one letter d such that $vd \in \mathrm{Fact}(\mathrm{hpw}(w))$, or exist exactly two letters d_1, d_2 such that $vd_1, vd_2 \in \mathrm{Fact}(\mathrm{hpw}(w))$.*

Further, the latter case occurs if and only if v ends with exactly k letters c, with $k \leqslant n \leqslant (l+2)\cdot 2^{k-1} + k - 1$.

Proof. Observe the following easy to see corollary of the definition of $\mathrm{hpw}(w)$: for any occurrence of the letter c in $\mathrm{hpw}(w)$ such that both the letters preceding it and following it are $\neq c$, this letter c is necessarily followed by $\widetilde{w_0}$; for any occurrence of a sequence of two or more consecutive letters c in $\mathrm{hpw}(w)$, this sequence is followed by w_0.

Let $v = b_1 b_2 \dots b_n$. Assume that v ends with exactly k letters c, where $0 \leqslant k \leqslant n$.

Consider the case $k = 0$, that is, $b_n \neq c$. Since $(l+2) \cdot 2^{k-1} + k - 1 = (l+2) \cdot 2^{-1} + 0 - 1 = \frac{l}{2} < n$, we have to prove that in this case there is uniquely determined letter d such that $vd \in \mathrm{Fact}(\mathrm{hpw}(w))$. Since $n \geqslant l+3$, the letter c must occur in v; in fact, it must occur in $b_3 b_4 \dots b_n$. Let $b_t = c$ be the last occurrence of c in v, where $3 \leqslant t \leqslant n-1$. We thus have that $b_{t+1} b_{t+2} \dots b_n$ is a prefix of w_0 or $\widetilde{w_0}$. By the observation given above, we see that if $b_{t-1} = c$, then $b_{t+1} b_{t+2} \dots b_n$ is a prefix of w_0, while if $b_{t-1} \neq c$, then $b_{t+1} b_{t+2} \dots b_n$ is a prefix of $\widetilde{w_0}$. Both of these possibilities lead to conclusion that there is only one letter d such that $vd \in \mathrm{Fact}(\mathrm{hpw}(w))$: d is the letter that follows $b_{t+1} b_{t+2} \dots b_n$ in w_0, respectively $\widetilde{w_0}$, or, if $b_{t+1} b_{t+2} \dots b_n$ is equal to w_0 or $\widetilde{w_0}$, then $d = c$.

Let now $k = n$, that is, $v = c^n$. Since $(l+2) \cdot 2^{k-1} + k - 1 = (l+2) \cdot 2^{n-1} + n - 1 > n$, we have to prove that in this case there are exactly two letters d_1, d_2 such that $vd_1, vd_2 \in \mathrm{Fact}(\mathrm{hpw}(w))$. And indeed, the only two such letters are $d_1 = c$ and $d_2 = a_1$ (in case $a_1 \neq a_l$, there cannot be $d_2 = a_l$ because of $n > 1$ and the observation from the beginning of the proof).

Finally, let k be such that $1 \leqslant k \leqslant n-1$, that is, $b_n = b_{n-1} = \cdots = b_{n-k+1} = c$ and $b_{n-k} \neq c$. It is easy to see that the only two letters d_1, d_2 such that we could possibly have $vd_1, vd_2 \in \mathrm{Fact}(\mathrm{hpw}(w))$ are $d_1 = c$ and either $d_2 = a_1$ (in case $k > 1$) or $d_2 = a_l$ (in case $k = 1$). We first prove the following claim: if $n > l_{k-1} + 2k$, then we cannot have both $vc \in \mathrm{Fact}(\mathrm{hpw}(w))$ and $vd_2 \in \mathrm{Fact}(\mathrm{hpw}(w))$. This is proved by showing (under the assumption $n > l_{k-1} + 2k$) that if $vc \in \mathrm{Fact}(\mathrm{hpw}(w))$, then a particular letter in the word v cannot equal c, while if $vd_2 \in \mathrm{Fact}(\mathrm{hpw}(w))$, then the same letter must equal c.

Assume $n > l_{k-1} + 2k$ and $vc \in \mathrm{Fact}(\mathrm{hpw}(w))$. Wherever the word $b_{n-k+1} b_{n-k+2} \dots b_n c = c^{k+1}$ is positioned in $\mathrm{hpw}(w)$, it is clearly a part of a middle segment c^s of a copy of $w_s = w_{s-1} c^s \widetilde{w_{s-1}}$ for some $s \geqslant k+1$. We have that $w_k = w_{k-1} c^k \widetilde{w_{k-1}}$ is a prefix of w_{s-1}, and thus $\widetilde{w_k} = w_{k-1} c^k \widetilde{w_{k-1}}$ is a suffix of $\widetilde{w_{s-1}} = w_{s-1}$. Since $n - l_{k-1} - 2k \geqslant 1$, it follows that $b_{n - l_{k-1} - 2k} \neq c$ (see Figure 1).

Still assuming $n > l_{k-1} + 2k$, further assume that now $vd_2 \in \mathrm{Fact}(\mathrm{hpw}(w))$ (where $d_2 = a_1$ if $k > 1$, and $d_2 = a_l$ if $k = 1$). Wherever the word $b_{n-k} b_{n-k+1} \dots b_n d_2 = b_{n-k} c^k d_2$ is positioned in $\mathrm{hpw}(w)$, it is clearly contained in a copy of w_k. We claim that there is a sequence of at least $k+1$ consecutive letters c immediately preceding this copy of w_k. There exists a copy of

$$\begin{array}{rl|c|c|l}
 & \multicolumn{3}{c}{\overbrace{\hphantom{b_1 \dots b_{n-l_{k-1}-2k} \quad b_{n-l_{k-1}-2k+1} \dots b_{n-k-1} b_{n-k} \quad c^k}}^{v}} & \\
vc: & b_1 \dots b_{n-l_{k-1}-2k} & b_{n-l_{k-1}-2k+1} \dots b_{n-k-1} b_{n-k} & c^k & c \\
\mathrm{hpw}(w): & \dots\dots w_{k-1} & c^k \widetilde{w_{k-1}} & c^k & c^{s-k} \widetilde{w_{s-1}} \dots \\
 & \multicolumn{2}{c}{\underbrace{\hphantom{\dots w_{k-1} \quad c^k \widetilde{w_{k-1}} \qquad\qquad}}_{w_{s-1}}} & &
\end{array}$$

Figure 1: $b_{n-l_{k-1}-2k} \neq c$.

$w_{k+1} = w_k c^{k+1} \widetilde{w_k} = w_k c^{k+1} w_k$ such that the considered copy of w_k coincides either with a prefix or with a suffix of this copy of w_{k+1}. In the latter case, it is preceded by c^{k+1}, as claimed. Thus, assume the former case. The considered copy of w_{k+1} now coincides either with a prefix or with a suffix of a copy of $w_{k+2} = w_{k+1} c^{k+2} w_{k+1}$. In the latter case, it is preceded by c^{k+2}, and thus its prefix w_k also is preceded by c^{k+2}, as claimed. Thus, assume again the former case. The copy of $w_k = w_{k-1} c^k \widetilde{w_{k-1}}$ we begin with cannot be positioned at the beginning of $\mathrm{hpw}(w)$, since there should be at least $n - k > l_{k-1} + k > l_{k-1}$ letters before c^k. Therefore, if the procedure above is repeated, it eventually happens that for some $r \geqslant k$ the considered copy of w_k coincides with a prefix of a copy of w_r that in turn coincides with a suffix of a copy of $w_{r+1} = w_r c^{r+1} w_r$. Thus, the considered copy of w_k is preceded by c^{r+1}, which proves the claim. Therefore, $b_{n-l_{k-1}-2k} = c$ (see Figure 2).

$$\begin{array}{rl|c|c|l}
 & \multicolumn{3}{c}{\overbrace{\hphantom{b_1 \dots b_{n-l_{k-1}-2k} \quad b_{n-l_{k-1}-2k+1} \dots b_{n-k-1} b_{n-k} \quad c^k}}^{v}} & \\
vd_2: & b_1 \dots b_{n-l_{k-1}-2k} & b_{n-l_{k-1}-2k+1} \dots b_{n-k-1} b_{n-k} & c^k & d_2 \\
\mathrm{hpw}(w): & \dots w_r c^{r+1-k} & c^k w_{k-1} & c^k & \widetilde{w_{k-1}} \dots\dots \\
 & & \multicolumn{3}{r}{\underbrace{\hphantom{w_{k-1} \quad c^k \quad \widetilde{w_{k-1}} \dots\dots}}_{w_r}}
\end{array}$$

Figure 2: $b_{n-l_{k-1}-2k} = c$.

Summing the results, we have proved that if

$$n > l_{k-1} + 2k = (l+2) \cdot 2^{k-1} - (k-1) - 2 + 2k = (l+2) \cdot 2^{k-1} + k - 1,$$

then not both vd_1, vd_2 can belong to $\mathrm{Fact}(\mathrm{hpw}(w))$. In order to finish the proof, it is enough to prove the reverse direction for $1 \leqslant k \leqslant n-1$. Let $n \leqslant l_{k-1} + 2k = (l+2) \cdot 2^{k-1} + k - 1$. There does not exist such n for $k = 1$, since otherwise we would have $n \leqslant (l+2) \cdot 2^{1-1} + 1 - 1 = l + 2$, contradicting the assumption $n \geqslant l + 3$. Thus, we are left to check the case $2 \leqslant k \leqslant n - 1$. In this case we have $\widetilde{w_{k-1}} = w_{k-1}$; therefore, v is a suffix of $c^k w_{k-1} c^k$, and $vc, va_1 \in \mathrm{Fact}(\mathrm{hpw}(w))$ (Figures 1 and 2 could again help visualizing these conclusions). This completes the proof. ■

Lemma 4.4. *For each $n \geqslant l+3$ we have:*

$$T_{\mathrm{hpw}(w)}(n) = 0.$$

Proof. Each $v \in \mathrm{Pal}(\mathrm{hpw}(w))$ such that $|v| \geqslant l+3$ clearly does not belong to $\mathrm{Pal}(w)$. Let $n \geqslant l+3$ be given, and let

$$\begin{aligned} A &= \{k \geqslant 1 : k \leqslant n \leqslant (l+2) \cdot 2^k + k \text{ and } k \equiv n \pmod 2\}, \\ B &= \{k \geqslant 1 : k \leqslant n+1 \leqslant (l+2) \cdot 2^k + k \text{ and } k \equiv n+1 \pmod 2\}, \\ C &= \{k \geqslant 1 : k-1 \leqslant n \leqslant (l+2) \cdot 2^k + k\}. \end{aligned}$$

We claim that $A \cap B = \varnothing$ and $A \cup B = C$. It is easy to see that $A \cap B = \varnothing$ and $A, B \subseteq C$, and thus we are left to prove that $C \subseteq A \cup B$. Let $k \in C$. If $k \equiv n \pmod 2$, then $k-1 \neq n$, and thus $k-1 < n$, that is, $k \leqslant n$; therefore, $k \in A$. If $k \equiv n+1 \pmod 2$, then $n \neq (l+2) \cdot 2^k + k$, and thus $n < (l+2) \cdot 2^k + k$, that is, $n+1 \leqslant (l+2) \cdot 2^k + k$; therefore, $k \in B$.

By Lemma 4.2, we now deduce:

$$P_{\mathrm{hpw}(w)}(n) + P_{\mathrm{hpw}(w)}(n+1) = |A| + |B| = |C|.$$

By Lemma 4.3, we have:

$$C_{\mathrm{hpw}(w)}(n+1) - C_{\mathrm{hpw}(w)}(n) = |\{k \geqslant 0 : k \leqslant n \leqslant (l+2) \cdot 2^{k-1} + k - 1\}|.$$

Clearly, the set $\{k \geqslant 0 : k \leqslant n \leqslant (l+2) \cdot 2^{k-1} + k - 1\}$ is an interval, say $[k_{\min}, k_{\max}]$. Actually, we have $k_{\max} = n$. The set C is also an interval, and we claim that $C = [k_{\min} - 1, n+1]$. It is easy to see that $n+1 \in C$ and $n+2 \notin C$. Let us show the other bound. Since $(l+2) \cdot 2^{1-1} + 1 - 1 = l+2 < n$, we have $k_{\min} \geqslant 2$. Therefore, $k_{\min} - 1 \geqslant 1$. From $k_{\min} \leqslant n$ and $n \leqslant (l+2) \cdot 2^{k_{\min}-1} + k_{\min} - 1$ we deduce $k_{\min} - 1 < n$ and $k_{\min} - 1 \in C$. Suppose $k_{\min} - 2 \in C$. We have $k_{\min} - 1 \leqslant n - 1 < n$. Further, from the supposed $k_{\min} - 2 \in C$ it follows that $n \leqslant (l+2) \cdot 2^{k_{\min}-2} + k_{\min} - 2$. Therefore, $k_{\min} - 1 \in [k_{\min}, n]$, which is a clear contradiction, and thus $k_{\min} - 2 \notin C$. This proves $C = [k_{\min} - 1, n+1]$.

Finally,

$$\begin{aligned} T_{\mathrm{hpw}(w)}(n) &= C_{\mathrm{hpw}(w)}(n+1) - C_{\mathrm{hpw}(w)}(n) + 2 - P_{\mathrm{hpw}(w)}(n) - P_{\mathrm{hpw}(w)}(n+1) \\ &= |[k_{\min}, n]| + 2 - |[k_{\min} - 1, n+1]| \\ &= (n - k_{\min} + 1) + 2 - (n + 1 - (k_{\min} - 1) + 1) = 0, \end{aligned}$$

which was to be proved. ∎

Lemma 4.5. *We have:*

$C_{\mathrm{hpw}(w)}(l+3) = 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : |v| \leqslant l+3\}| - P_{\mathrm{hpw}(w)}(l+3) - 2.$

Proof. We begin by listing all the factors of $\mathrm{hpw}(w)$ of length $l+3$. However, since $\mathrm{Fact}(\mathrm{hpw}(w))$ is closed under reversal, we shall not include both a factor and its reversal in the list, but choose only one representative for each such pair. We claim that the left column of Table 1 presents the described list.

$u \in \mathrm{Fact}(\mathrm{hpw}(w)),\ \lvert u\rvert = l+3$	the longest prefix or suffix of u from $\mathrm{Pal}(\mathrm{hpw}(w)) \setminus (\mathrm{Pal}(w) \cup \{c\})$
wca_la_{l-1}	$a_{l-1}a_lca_la_{l-1}$
$cwca_l$	a_lca_l
$ccwc$	cc
$cccw$	ccc
a_1ccw	a_1cca_1
$a_{l-s}\dots a_{l-1}a_lca_la_{l-1}\dots a_s$ $(l-2 \geqslant s \geqslant \lceil \frac{l}{2} \rceil)$	$a_s\dots a_{l-1}a_lca_la_{l-1}\dots a_s$
$a_{l+3-t}\dots a_2a_1c^t$ $(4 \leqslant t \leqslant l+2)$	c^t
$a_{l+2-t}\dots a_2a_1c^ta_1$ $(3 \leqslant t \leqslant l+1)$	$a_1c^ta_1$
$a_{l+3-s-t}\dots a_2a_1c^ta_1a_2\dots a_s$ $(2 \leqslant s \leqslant \lfloor \frac{l+1}{2} \rfloor,\ 2 \leqslant t \leqslant l+3-2s)$	$a_s\dots a_2a_1c^ta_1a_2\dots a_s$
c^{l+3}	c^{l+3}

Table 1: Factors and their longest palindromic prefixes or suffixes.

The list is compiled by the following approach:

- We first enumerate all $u \in \mathrm{Fact}(\mathrm{hpw}(w))$, $|u| = l+3$ such that $w \in \mathrm{Fact}(u)$. Depending on whether w begins with the first, the second, the third or the fourth letter of u, we easily see that in all of these cases but the last one the other characters are uniquely determined, while in the last case there are exactly two possibilities. These five factors are shown in the first group. These factors also stand as the representative of factors u such that $\widetilde{w} \in \mathrm{Fact}(u)$.
- We now enumerate all $u \in \mathrm{Fact}(\mathrm{hpw}(w))$, $|u| = l+3$, such that u ends with a prefix of $\widetilde{w}$, say $a_la_{l-1}\dots a_s$. We see that in this case

$u = a_{l-s} \dots a_{l-1} a_l c a_l a_{l-1} \dots a_s$ (the "left end" is calculated so that $(l-(l-s)+1)+1+(l-s+1) = l+3)$. Since we require $w, \widetilde{w} \notin \mathrm{Fact}(u)$ (in order to avoid repeating a factor already included in the first group), we deduce the bounds $s \geqslant 2$ and $s \leqslant l-2$. Furthermore, since reversals of factors from this group are of the same form, in order to avoid repeating we require $|a_{l-s} \dots a_{l-1} a_l| \geqslant |a_l a_{l-1} \dots a_s|$, that is, $s \geqslant l-s$, that is, $s \geqslant \lceil \frac{l}{2} \rceil$. Altogether, the bounds are $l-2 \geqslant s \geqslant \lceil \frac{l}{2} \rceil$ (for $l = 3$ and $l = 2$ this group is empty).

- In the third group we enumerate all the considered factors u that end with c^t (where t is maximal possible), but $u \neq c^{l+3}$. It cannot be $t = 1, 2, 3$, since u would contain w or $\widetilde{w}$. Therefore, $u = a_{l+3-t} \dots a_2 a_1 c^t$ with bounds $4 \leqslant t \leqslant l+2$.

- We now check what are the possibilities when u ends with a prefix of w, say $a_1 a_2 \dots a_s$. In fact, since the case $s = 1$ is slightly different from the cases where $s > 1$ (the lower bounds for t differ), we treat them separately. Thus, in this group we let $u = a_{l+2-t} a_{l+1-t} \dots a_1 c^t a_1$. The bounds are $l-1 \geqslant l+2-t \geqslant 1$, that is, $3 \leqslant t \leqslant l+1$.

- Let now $u = a_{l+3-s-t} \dots a_2 a_1 c^t a_1 a_2 \dots a_s$, $s \geqslant 2$. The bound $t \geqslant 2$ is clear. In order to avoid including both a factor and its reversal, we require $|a_{l+3-s-t} \dots a_2 a_1| \geqslant |a_1 a_2 \dots a_s|$, that is, $l+3-s-t \geqslant s$, that is, $t \leqslant l+3-2s$. For a fixed s, we have the bounds $l-1 \geqslant l+3-s-t \geqslant 1$, that is, $4-s \leqslant t \leqslant l+2-s$. Since $4-s \leqslant 2$ and $l+3-2s \leqslant l+2-s$, the bounds for t are $2 \leqslant t \leqslant l+3-2s$. Considering the bounds for s, we already have $s \geqslant 2$, and the upper bound follows from the requirement that t exists: $2 \leqslant l+3-2s$, that is, $s \leqslant \lfloor \frac{l+1}{2} \rfloor$.

- Finally, there is one more factor not included so far: c^{l+3}.

For each of the enumerated factors, we find out that either its longest palindromic prefix or longest palindromic suffix, but not both, belongs to $\mathrm{Pal}(\mathrm{hpw}(w)) \setminus (\mathrm{Pal}(w) \cup \{c\})$. These prefixes and suffixes are shown in the right column of Table 1. We claim that such a correspondence is in fact a bijection between the left column and the set $\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : 2 \leqslant |v| \leqslant l+3\}$. Therefore, it is enough to check whether each v from this set appears exactly once in the right column.

We shall enumerate these palindromes by ordering them with respect to the number of consecutive occurrences of the letter c in the middle (by

Lemma 4.2, this parameter and the length uniquely determine palindrome). If there is one letter c in the middle, the palindromes of length 3 and 5 are in the first group, while the palindromes of length 7 and more are in the second group. If there are two letters c in the middle, the palindromes of length 2 and 4 are in the first group, while the palindromes of length 6 and more are in the fifth group (for $t = 2$, s ranges from 2 to $\lfloor \frac{l+1}{2} \rfloor$, and thus the length of the observed palindromes takes all the even values from 6 to $l+3$ or $l+2$, depending on the parity of l). If there are three letters c in the middle, the palindrome of length 3 is in the first group, the palindrome of length 5 is in the fourth group, while the palindromes of length 7 and more are in the fifth group (for $t = 3$, s ranges from 2 to the largest value meeting the requirement $3 \leqslant l + 3 - 2s$, which is $\lfloor \frac{l}{2} \rfloor$, and thus the length of observed palindromes takes all the odd values from 7 to $l+3$ or $l+2$). Continuing in this manner, we enumerate all the considered palindromes, and prove the claim.

Therefore, there are $|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : 2 \leqslant |v| \leqslant l+3\}|$ factors in the left column. Since for each pair $\{u, \widetilde{u}\}$ of factors of $\mathrm{hpw}(w)$ of length $l+3$ only one representative is included in the left column, we have that the number of factors of $\mathrm{hpw}(w)$ of length $l+3$ equals the number of palindromic factors of $\mathrm{hpw}(w)$ of length $l+3$ plus twice the number of nonpalindromic factors in the left column. In short:

$$
\begin{aligned}
&C_{\mathrm{hpw}(w)}(l+3) \\
&\quad = P_{\mathrm{hpw}(w)}(l+3) \\
&\qquad + 2\big(|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : 2 \leqslant |v| \leqslant l+3\}| - P_{\mathrm{hpw}(w)}(l+3)\big) \\
&\quad = 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : 2 \leqslant |v| \leqslant l+3\}| - P_{\mathrm{hpw}(w)}(l+3) \\
&\quad = 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : |v| \leqslant l+3\} \setminus \{c\}| - P_{\mathrm{hpw}(w)}(l+3) \\
&\quad = 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : |v| \leqslant l+3\}| - P_{\mathrm{hpw}(w)}(l+3) - 2,
\end{aligned}
$$

which was to be proved. ■

Proof of Theorem 4.1. We have:

$$
\begin{aligned}
\sum_{n=0}^{\infty} T_{\mathrm{hpw}(w)}(n) &\overset{\text{L4.4}}{=} \sum_{n=0}^{l+2} T_{\mathrm{hpw}(w)}(n) \\
&= \sum_{n=0}^{l+2} (C_{\mathrm{hpw}(w)}(n+1) - C_{\mathrm{hpw}(w)}(n) + 2 - P_{\mathrm{hpw}(w)}(n) - P_{\mathrm{hpw}(w)}(n+1)) \\
&= \sum_{n=0}^{l+2} C_{\mathrm{hpw}(w)}(n+1) - \sum_{n=0}^{l+2} C_{\mathrm{hpw}(w)}(n) + 2(l+3) \\
&\qquad\qquad - \sum_{n=0}^{l+2} P_{\mathrm{hpw}(w)}(n) - \sum_{n=0}^{l+2} P_{\mathrm{hpw}(w)}(n+1) \\
&= C_{\mathrm{hpw}(w)}(l+3) - C_{\mathrm{hpw}(w)}(0) + 2(l+3) \\
&\qquad\qquad - 2\sum_{n=0}^{l+3} P_{\mathrm{hpw}(w)}(n) + P_{\mathrm{hpw}(w)}(0) + P_{\mathrm{hpw}(w)}(l+3) \\
&= C_{\mathrm{hpw}(w)}(l+3) - 1 + 2(l+3) - 2\sum_{n=0}^{l+3} P_{\mathrm{hpw}(w)}(n) + 1 + P_{\mathrm{hpw}(w)}(l+3) \\
&\overset{\text{L4.5}}{=} 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : |v| \leqslant l+3\}| - P_{\mathrm{hpw}(w)}(l+3) - 2 \\
&\qquad\qquad + 2l + 6 - 2\sum_{n=0}^{l+3} P_{\mathrm{hpw}(w)}(n) + P_{\mathrm{hpw}(w)}(l+3) \\
&= 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : |v| \leqslant l+3\}| + 2l + 4 - 2\sum_{n=0}^{l+3} P_{\mathrm{hpw}(w)}(n) \\
&= 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) \setminus \mathrm{Pal}(w) : |v| \leqslant l+3\}| + 2l + 4 \\
&\qquad\qquad - 2|\{v \in \mathrm{Pal}(\mathrm{hpw}(w)) : |v| \leqslant l+3\}| \\
&= 2l + 4 - 2|\{v \in \mathrm{Pal}(w) : |v| \leqslant l+3\}| = 2l + 4 - 2|\mathrm{Pal}(w)| \\
&= 2(D(w)+1) \overset{\text{T3.2}}{=} 2D(\mathrm{hpw}(w)),
\end{aligned}
$$

which was to be proved. ■

Note. After this paper was submitted, a proof of the full Brlek-Reutenauer conjecture, by Balková, Pelantová and Starosta, appeared in [4].

5 Longest palindromic suffixes of factors of a highly potential word

It was claimed in [3, Theorem 5.7] that for any infinite word u having the set of factors closed under reversal and containing infinitely many palindromes the following statements are equivalent:

(a) $D(u) < \infty$;

(b) there exists $H \in \mathbb{N}_0$ such that the longest palindromic suffix of any $v \in \mathrm{Fact}(u)$, of length $|v| \geqslant H$, occurs in v exactly once.

A mistake in the proof was spotted in [5], and a counterexample was constructed (which is actually not a highly potential word; this can be seen by, e.g., noting that the constructed counterexample does not have any two occurrences of the same letter next to each other). We hereby show that actually all the highly potential words are counterexamples to the above claim. Since each highly potential word has the set of factors closed under reversal, contains infinitely many palindromes and is of finite defect, it is enough to prove:

Theorem 5.1. *Each highly potential word* $\mathrm{hpw}(w)$ *contains arbitrarily long factors* v *such that the longest palindromic suffix of* v *occurs in* v *more than once.*

Proof. For each $i \geqslant 2$ the word $wc = w_0c$ is a prefix of the word $w_{i-1} = \widetilde{w_{i-1}}$. Therefore, $c^iwc \in \mathrm{Fact}(w_i) \subseteq \mathrm{Fact}(\mathrm{hpw}(w))$. Since the letter c does not occur in the word w, and $w \neq \widetilde{w}$, the longest palindromic suffix of the word c^iwc is clearly only the letter c, having $i+1$ occurrences in c^iwc. ∎

6 Highly potential word fixed by a morphism

As mentioned in Section 1, Brlek and Reutenauer showed that Conjecture 2.6 holds for all fixed points of nonidentical morphisms, under the conjecture that there does not exist an aperiodic word of finite positive defect that is a fixed point of a nonidentical morphism. However, in this section we construct a highly potential word that is a fixed point of a nonidentical morphism, thus showing that the conjecture assumed by Brlek and Reutenauer is false.

Theorem 6.1. *Let $\Sigma = \{a, b, c\}$, let the morphism φ be defined by $\varphi(a) = abcbac$, $\varphi(b) = \varepsilon$, $\varphi(c) = c$, and let $w = ab$. Then:*

$$\varphi(\mathrm{hpw}(w)) = \mathrm{hpw}(w).$$

Proof. It is enough to prove that for each $i \in \mathbb{N}_0$ the word $\varphi(w_i)$ is a prefix of $\mathrm{hpw}(w)$. By induction on i, we shall prove that, for each $i \in \mathbb{N}_0$, $\varphi(w_i) = w_{i+1}c$ (and since this is a prefix of w_{i+2} and therefore also a prefix of $\mathrm{hpw}(w)$, the proof would thus be completed). We have:

$$w_0 = ab;$$

$$w_1 = abcba;$$

$$w_2 = abcbaccabcba.$$

For $i = 0$ we have:

$$\varphi(w_0) = \varphi(ab) = abcbac = w_1c.$$

For $i = 1$ we have:

$$\varphi(w_1) = \varphi(abcba) = abcbac\,c\,abcbac = w_2c.$$

For $i \geqslant 2$ we have:

$$\begin{aligned}\varphi(w_i) &= \varphi(w_{i-1}c^i\widetilde{w_{i-1}}) = \varphi(w_{i-1}c^iw_{i-1}) = \varphi(w_{i-1})\varphi(c)^i\varphi(w_{i-1})\\ &= w_ic\,c^i\,w_ic = w_ic^{i+1}w_ic = w_{i+1}c,\end{aligned}$$

which was to be proved. ■

Notes. (1) The reader may check that the highly potential word considered in the previous theorem (which is the essentially unique highly potential word generated by a word of length 2) is also fixed by a *nonerasing* morphism φ (that is: a morphism that maps none of the letters to ε) defined by $\varphi(a) = \varphi(b) = abcbacc$ and $\varphi(c) = c$.

(2) Neither the morphism φ from Theorem 6.1 nor the one from the previous note is primitive. Therefore, the original conjecture by Blondin-Massé et al., mentioned in the Introduction, is still standing.

Acknowledgments

The author would like to thank the two anonymous referees for their useful comments, which helped to improve the content of this paper.

The research was supported by the Ministry of Science and Technological Development of Serbia (project 174006).

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